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Hamilton-Jacobi Functional Theory for the Integration of Classical Field Equations†

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Abstract

By employing the terminology of functional differential calculus, Hamilton-Jacobi theory is extended to apply to classical field equations. It is shown that an asymptotic solution to the Hamilton-Jacobi functional differential equation provides an asymptotic general solution to the associated nonlinear classical field equations.

The usefulness of Hamilton-Jacobi (H-J) theory for interacting particle systems in classical mechanics‡ is well known. The purpose of this note is to exhibit appropriately extended H-J theory for classical field equations. It is shown that an asymptotic general solution to nonlinear classical field equations follows from an asymptotic solution to the associated H-J functional differential equation.

From the Hamiltonian functional $H = H[\phi, \pi]$ one obtains the canonical field equations§

$$\partial\phi(\mathbf{x}, t)/\partial t = \delta H/\delta\pi(\mathbf{x}, t), \qquad \partial\pi(\mathbf{x}, t)/\partial t = -\delta H/\delta\phi(\mathbf{x}, t) \tag{1}$$

and the H-J functional differential equation

$$H\left[\phi, \frac{\delta S}{\delta\phi}\right] + \frac{\partial S}{\partial t} = 0 \tag{2}$$

for the principal functional $S = S[\phi, \alpha, t]$, where $\alpha = \alpha(\mathbf{x})$ is a disposable 'function of integration' that does not depend on t. By paraphrasing the

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‡ See, for example, Goldstein, H. (1953). *Classical Mechanics*, pp. 273–288. Addison-Wesley Publishing Co., Inc., Cambridge, Mass.; a more rigorous and detailed mathematical discussion of Hamilton-Jacobi theory for particle mechanics has been given by Forsyth, A. R. (1959). *Theory of Differential Equations*, Vol. V, pp. 371–390. Dover Publications, Inc., New York.

§ The notation is standard, conforming with Rosen, G. (1969). *Formulations of Classical and Quantum Dynamical Theory*, pp. 57–59, 96–100. Academic Press, Inc., New York.

argument employed in classical mechanics,† it is easy to prove that the field

$$\beta(\mathbf{x}) \equiv \delta S/\delta\alpha(\mathbf{x}) \tag{3}$$

is also independent of t; the principal functional S generates a canonical transformation from the time-dependent canonical field variables ϕ and π to the time-independent canonical field variables β and α. Moreover, since the Hamiltonian functional is not supposed to display an explicit dependence on t, we have the form $S = W - Et$ satisfying equation (2) with the energy $E = E[\alpha]$ a positive functional of α alone (independent of ϕ, t) and the functional $W = W[\phi, \alpha]$ a solution to the equation

$$H\left[\phi, \frac{\delta W}{\delta\phi}\right] = E \tag{4}$$

It is convenient to require the disposable function of integration α to be nonnegative, $\alpha = \alpha(\mathbf{x}) \geqslant 0$, and such that

$$\int \alpha(\mathbf{x})\, d^3 x = E \tag{5}$$

for then (3) yields the simple relation

$$\beta(x) + t = \frac{\delta W}{\delta\alpha(\mathbf{x})} \tag{6}$$

The general solution to equation (4) is unique to within a trivial additive functional of α alone (independent of ϕ) which merely alters the definition of β in (6). Having solved the functional differential equation (4) for $W = W[\phi, \alpha]$, we obtain a general solution to the field equations (1) by inverting (6) to get $\phi = \phi(\mathbf{x}, t)$ explicitly in terms of the disposable functions of integration α and β.

To illustrate the application of this extended H-J theory for the integration of classical field equations, let us consider the model Hamiltonian functional

$$H = \int \tfrac{1}{2}(\pi^2 + |\nabla\phi|^2 + m^2\phi^2)\, d^3 x \qquad [m^2 \geqslant 0] \tag{7}$$

for which the implied canonical field equations (1) combine to produce the linear wave equation

$$\partial^2\phi/\partial t^2 - \nabla^2\phi + m^2\phi = 0 \tag{8}$$

In the case of (7), equation (4) takes the form

$$\int \frac{1}{2}\left[\left(\frac{\delta W}{\delta\phi}\right)^2 + |\nabla\phi|^2 + m^2\phi^2\right] d^3 x = E \tag{9}$$

† See footnote on p. 281.

The latter functional differential equation yields the solution

$$W = W_0 \equiv \sum_{n=0}^{\infty} \left\{ \tfrac{1}{2}\xi_n \sqrt{[2\epsilon_n - (\lambda_n + m^2)\xi_n{}^2]} + \frac{\epsilon_n}{\sqrt{(\lambda_n + m^2)}} \sin^{-1}\left[\sqrt{\left(\frac{\lambda_n + m^2}{2\epsilon_n}\right)}\,\xi_n\right]\right\} \tag{10}$$

for $\xi_n{}^2 \leqslant 2\epsilon_n/(\lambda_n + m^2)$ where

$$\xi_n \equiv \int \phi u_n \, d^3x, \qquad \epsilon_n \equiv \left(\int \alpha^{1/2} u_n \, d^3x\right)^2 \tag{11}$$

with the u_n's a complete orthonormal set of real eigenfunctions of the operator $(-\nabla^2)$,

$$-\nabla^2 u_n = \lambda_n u_n, \qquad u_n{}^* = u_n, \qquad \lambda_n{}^* = \lambda_n \tag{12}$$

$$\sum_{n=0}^{\infty} u_n(\mathbf{x})\, u_n(\mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \qquad \int u_n u_{n'} \, d^3x = \delta_{nn'}$$

We verify that (10) satisfies the functional differential equation (9) by making use of the chain-rule formula

$$\frac{\delta W}{\delta\phi(\mathbf{x})} = \frac{\delta W_0}{\delta\phi(\mathbf{x})} = \sum_{n=0}^{\infty} \frac{\delta\xi_n}{\delta\phi(\mathbf{x})} \frac{\partial W_0}{\partial \xi_n} = \sum_{n=0}^{\infty} u_n(\mathbf{x}) \frac{\partial W_0}{\partial \xi_n} \tag{13}$$

which implies that

$$\int \left(\frac{\delta W}{\delta\phi}\right)^2 d^3x = \sum_{n=0}^{\infty} \left(\frac{\partial W_0}{\partial \xi_n}\right)^2 = \sum_{n=0}^{\infty} [2\epsilon_n - (\lambda_n + m^2)\xi_n{}^2] \tag{14}$$

and by using (11) and (12) to write

$$\int [|\nabla\phi|^2 + m^2\phi^2] \, d^3x = \sum_{n=0}^{\infty} (\lambda_n + m^2)\xi_n{}^2 \tag{15}$$

and

$$E = \int \alpha \, d^3x = \sum_{n=0}^{\infty} \epsilon_n \tag{16}$$

The functional derivative of (10) with respect to $\alpha(x)$ is

$$\frac{\delta W}{\delta\alpha(\mathbf{x})} = \frac{\delta W_0}{\delta\alpha(\mathbf{x})} = \sum_{n=0}^{\infty} \frac{\delta\epsilon_n}{\delta\alpha(\mathbf{x})} \frac{\partial W_0}{\partial\epsilon_n} = \alpha(\mathbf{x})^{-1/2} \sum_{n=0}^{\infty} \epsilon_n^{1/2} u_n(\mathbf{x}) \frac{\partial W_0}{\partial\epsilon_n}$$

$$= \alpha(\mathbf{x})^{-1/2} \sum_{n=0}^{\infty} \left\{ u_n(\mathbf{x}) \sqrt{\left(\frac{\epsilon_n}{\lambda_n + m^2}\right)} \sin^{-1}\left[\sqrt{\left(\frac{\lambda_n + m^2}{2\epsilon_n}\right)}\,\xi_n\right]\right\} \tag{17}$$

and thus it follows from (6) that

$$\int (\beta + t)\,\alpha^{1/2} u_n \, d^3x = \int \frac{\delta W}{\delta \alpha} \alpha^{1/2} u_n \, d^3x$$
$$= \sqrt{\left(\frac{\epsilon_n}{\lambda_n + m^2}\right)} \sin^{-1}\left[\sqrt{\left(\frac{\lambda_n + m^2}{2\epsilon_n}\right)}\, \xi_n\right] \tag{18}$$

or equivalently that

$$\xi_n = \sqrt{\left(\frac{2\epsilon_n}{\lambda_n + m^2}\right)} \sin\left[\sqrt{(\lambda_n + m^2)}\, t + \theta_n\right] \tag{19}$$

with the phase constants defined as

$$\theta_n \equiv \sqrt{\left(\frac{\lambda_n + m^2}{\epsilon_n}\right)} \int \beta \alpha^{1/2} u_n \, d^3x \tag{20}$$

Hence, a general solution to the linear wave equation (8) is obtained from (19) by recalling formulas (11) and (12),

$$\phi = \phi(\mathbf{x}, t) = \sum_{n=0}^{\infty} \xi_n u_n(\mathbf{x})$$
$$= \sum_{n=0}^{\infty} \sqrt{\left(\frac{2\epsilon_n}{\lambda_n + m^2}\right)} \{\sin\left[\sqrt{(\lambda_n + m^2)}\, t + \theta_n\right]\} u_n(\mathbf{x}) \tag{21}$$

To illustrate the usefulness of the extended H-J integration theory for treating nonlinear classical field equations not amenable to more direct methods of solution, let us consider the model Hamiltonian functional

$$H = \int \tfrac{1}{2}(\pi^2 + |\nabla\phi|^2 + m^2\phi^2 - g\phi^6)\, d^3x \qquad [m^2 \geqslant 0, g > 0] \tag{22}$$

for which the implied canonical field equations (1) combine to produce the nonlinear wave equation†

$$\partial^2\phi/\partial t^2 - \nabla^2\phi + m^2\phi - 3g\phi^5 = 0 \tag{23}$$

In the case of (22), equation (4) takes the form

$$\int \frac{1}{2}\left[\left(\frac{\delta W}{\delta \phi}\right)^2 + |\nabla\phi|^2 + m^2\phi^2 - g\phi^6\right] d^3x = E \tag{24}$$

† For a study of certain solutions to the $m^2 = 0$ specialization of equation (23), see Rosen, G. (1965). *Journal of Mathematical Physics*, **6**, 1269; Rosen, G. (1967). *Journal of Mathematical Physics*, **8**, 573; Derrick, G. H. and Kay-Kong, W. (1968). *Journal of Mathematical Physics*, **9**, 232; Pinski, G. (1968). *Journal of Mathematical Physics*, **9**, 1323.

For ϕ^2 small compared to $g^{-1/2}(m^2+\phi^{-2}|\nabla\phi|^2)^{1/2}$ the latter functional differential equation yields the asymptotic solution

$$W = W_0 + \frac{g}{14}\int (2\alpha)^{-1/2}\phi^7[1+\alpha^{-1}0(m^2\phi^2+|\nabla\phi|^2)]\,d^3x \tag{25}$$

where W_0 is given by the final member in (10). On the other hand, for ϕ^2 large compared to $g^{-1/2}(m^2+\phi^{-2}|\nabla\phi|^2)^{1/2}$ we find the asymptotic solution

$$\begin{aligned} W = \int [&\tfrac{1}{4}g^{1/2}\phi^4 - \tfrac{1}{2}g^{-1/2}m^2(\ln|\phi|) - \tfrac{1}{8}g^{-1/2}\phi^{-2}|\nabla\phi|^2 \\ &-\tfrac{1}{2}g^{-1/2}\alpha\phi^{-2} + g^{-3/2}0(m^4\phi^{-4}+\phi^{-8}|\nabla\phi|^4)]\,d^3x \end{aligned} \tag{26}$$

Verification that (25) and (26) satisfy equation (24) asymptotically as indicated for small and large values of ϕ^2 follows immediately by computing the associated functional derivatives $\delta W/\delta\phi$ and substituting the latter expressions into (24). From (25), (6), and the final member in (17) we obtain the general perturbation solution to the nonlinear wave equation (23) for ϕ^2 small compared to $g^{-1/2}(m^2+\phi^{-2}|\nabla\phi|^2)^{1/2}$,

$$\begin{aligned} &\int \left(\beta + t + \frac{g}{14}(2\alpha)^{-3/2}\phi^7\right)\alpha^{1/2}u_n\,d^3x \\ &\qquad \doteq \sqrt{\left(\frac{\epsilon_n}{\lambda_n+m^2}\right)}\sin^{-1}\left[\sqrt{\left(\frac{\lambda_n+m^2}{2\epsilon_n}\right)}\xi_n\right] \end{aligned} \tag{27}$$

which yields a result of the form (21) if definition (20) for the phase constants is superseded by

$$\theta_n \equiv \sqrt{\left(\frac{\lambda_n+m^2}{\epsilon_n}\right)}\int\left[\beta\alpha^{1/2} + \frac{g\phi^7}{28\sqrt{(2)}\,\alpha}\right]u_n\,d^3x \tag{28}$$

for phase functions depending on t through the term in ϕ^7. With the disposable functions α and β prescribed, the phase functions (28) can be evaluated iteratively about the $g=0$ solution. From (26) and (6) we obtain the asymptotic form of the general 'runaway' solution to the nonlinear wave equation (23) for ϕ^2 large compared to $g^{-1/2}(m^2+\phi^{-2}|\nabla\phi|^2)^{1/2}$,

$$\phi \doteq \pm g^{-1/4}[-2(\beta+t)]^{-1/2} \tag{29}$$

attainable if $0 < -\beta \ll m^{-1}$ and becoming singular at $t = \inf_{\mathbf{x}}|\beta(\mathbf{x})|$. Note that both asymptotic solutions (27) and (29) can be sharpened by solving equation (24) for additional terms in the asymptotic series solutions (25) and (26). Likewise for other nonlinear classical field theories, an asymptotic solution to the H-J functional differential equation provides an asymptotic general solution to the associated field equations.